PERTURBATIVE CALCULATIONS AND THE TOP QUARK

BY

MARTIN C. SMITH

B.A., Northwestern University, 1992
M.S., University of Illinois at Chicago, 1993

THESIS

Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Physics in the Graduate College of the University of Illinois at Urbana-Champaign, 1998

Urbana, Illinois
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
THE GRADUATE COLLEGE

APRIL 1998

date

WE HEREBY RECOMMEND THAT THE THESIS BY

MARTIN C. SMITH

ENTITLED

PERTURBATIVE CALCULATIONS AND

THE TOP QUARK

BE ACCEPTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR

THE DEGREE OF

DOCTOR OF PHILOSOPHY

Director of Thesis Research

Head of Department

Committee on Final Examination†

V. R. Pedie

Chairperson

Brenna D. Sullivan

† Required for doctor's degree but not for master's.
Abstract

I calculate the $O(\alpha_s)$ QCD and the $O(\alpha_W m_t^2/M_W^2)$ Yukawa corrections to the production of a single top quark via the weak process $q\bar{q} \rightarrow t\bar{b}$ at the Fermilab Tevatron and the CERN Large Hadron Collider. An accurate calculation of the cross section is necessary in order to extract $|V_{tb}|$ from experiment.

Additionally, I show that the pole mass of a heavy quark is ambiguous by an amount proportional to $\Lambda_{\text{QCD}}$, and that this ambiguity is independent of the width of the heavy quark. The ambiguity is a consequence of the definition of the pole mass, and is unavoidable, even for a quark such as the top quark that decays more quickly than $\Lambda_{\text{QCD}}^{-1}$, and so might be expected to escape the effects of nonperturbative QCD.

Lastly, I show that the nonrelativistic perturbation series for heavy quarkonia energies diverges at large orders. This results in a perturbative ambiguity in the energy that scales as $e^{-2/\pi \Lambda_{\text{QCD}}}$, where $n$ is the principal quantum number and $a$ is the Bohr radius. This ambiguity is associated with a nonperturbative contribution to the energies arising from distances of order $\Lambda_{\text{QCD}}^{-1}$ and greater.
To my wife, Jamie
Acknowledgments

Thanks to Scott Willenbrock for his patient and constant tutelage. He has been a kind mentor and a consummate advisor to me, and his good taste in physics and facility in instruction are qualities to which I shall always aspire.

Thanks also to Tim Stelzer for many educating and entertaining discussions. He has earned my appreciation as a respected collaborator and a faithful friend.

I gratefully acknowledge the receipt of a GAANN fellowship under grant number DE-P200A10532 from the U. S. Department of Education. This work was also supported in part by Department of Energy grant DE-FG02-91ER40677.

Finally, thanks to my loving and gracious wife, Jamie, to whom this thesis is dedicated.
4 Nonperturbative Effects in Quarkonia .................. 35
4.1 A Renormalon-Inspired Look at Quarkonia .............. 35
4.2 Ambiguities and Nonperturbative Contributions .......... 40
5 Conclusions ............................................. 44
5.1 QCD and Yukawa Corrections to $q \bar{q} \rightarrow t \bar{b}$ ........ 44
5.2 Top-Quark Pole Mass .................................. 45
5.3 Nonperturbative Effects in Quarkonia .................... 46
Bibliography ................................................. 47
Curriculum Vitae ............................................. 51
List of Tables

1.1 The gauge bosons of the Standard Model with their masses and couplings. ........................................... 2

1.2 The matter content of the Standard Model, comprising three families of quarks and leptons. ................................. 3

2.1 Leading-order (LO) and next-to-leading-order (NLO) cross sections (pb) for \( q\bar{q} \rightarrow t\bar{b}, \bar{t}b \) at the Tevatron and the LHC for three different sets of parton distribution functions (PDFs). The NLO cross section including only the initial state (IS) correction is also given. ............................................................................................................. 22
List of Figures

2.1 Single-top-quark production via $q\bar{q} \rightarrow t\bar{b}$. ........................................ 7
2.2 Single-top-quark production via $W$-gluon fusion. ............................................. 9
2.3 $O(\alpha_s)$ correction to $q\bar{q} \rightarrow t\bar{b}$: (a)-(c) initial state, (d) final state. .... 11
2.4 Factorization-scale dependence of the leading-order (LO) and next-to-leading-order (NLO) cross sections for $q\bar{q} \rightarrow t\bar{b}, \bar{t}b$ at the Tevatron and the LHC. ............. 14
2.5 Renormalization-scale dependence of the leading-order (LO) and next-to-leading-order (NLO) cross sections for $q\bar{q} \rightarrow t\bar{b}, \bar{t}b$ at the Tevatron and the LHC. .......... 15
2.6 Differential cross section for $q\bar{q} \rightarrow t\bar{b}, \bar{t}b$ versus the mass of the virtual $s$-channel $W$ boson, at the Tevatron and the LHC. Both the leading-order (LO) and next-to-leading order (NLO) cross sections are shown, as well as the separate contributions from the initial-state (IS) and final-state (FS) corrections. ......................... 16
2.7 $O(\alpha_W m_t^2/M_W^2)$ corrections to $q\bar{q} \rightarrow t\bar{b}$. The dashed lines represent the Higgs boson and the unphysical scalar $W$ and $Z$ bosons in $R_t$ gauge: (a) wavefunction renormalization, (b) vertex correction. ........................................ 18
2.8 Fractional change in the total cross section for $q\bar{q} \rightarrow t\bar{b}, \bar{t}b$ due to the Yukawa correction vs. the Higgs-boson mass at the Tevatron and the LHC. .............. 20
2.9 Next-to-leading order cross section for $q \bar{q} \rightarrow t \bar{b}, t \bar{b}$ as a function of the top-quark mass. ................................................................. 23

3.1 A scattering amplitude consisting of two sub-amplitudes connected by a quark propagator. The external lines represent color-singlet asymptotic states. Such an amplitude is forbidden by color conservation. .............................................. 26

3.2 The production and decay of a top quark in (a) perturbation theory, and (b) nonperturbatively. ................................................................. 27

3.3 Diagrams contributing to the top-quark self-energy at leading order in $\alpha_s$ and $\alpha_W$. Fig. (a') replaces Fig. (a) when summing to all orders in $\beta_0 \alpha_s$. .................................................... 28

3.4 Diagrams contributing to the top-quark self-energy at leading order in $\alpha_s$, but to all orders in $\alpha_W$. (a') replaces (a) when summing to all orders in $\beta_0 \alpha_s$. .......................... 32

4.1 Quarkonium formation via the exchange of a gluon. The Coulomb potential is modified by the insertion of vacuum-polarization subdiagrams in the gluon propagator. 37

4.2 A higher-order contribution to the quarkonium energy from the emission and absorption of gluons. The insertion of vacuum-polarization subdiagrams in the gluon propagator leads to an infrared renormalon associated with the gluon condensate. 42
Chapter 1

Introduction

All of physicists’ current understanding of the fundamental constituents of matter and their interactions, with the exception of gravity, is integrated into the Standard Model of elementary particle physics. Over the past two decades, a multitude of experiments have confirmed the predictions of the Standard Model with amazing precision. Among the hundreds of observables measured in the tens of millions high-energy events collected at colliders worldwide, not a single significant deviation has been found. The recent discovery of the top quark [1] has completed the fermionic sector of the Standard Model, leaving only the Higgs boson undiscovered.

Despite its impressive successes, the Standard Model is not yet fully understood. There are many areas, notably those relating to the structure of electroweak symmetry breaking, which are as yet only poorly tested by experiment. Furthermore, the Standard Model’s relatively large number of free parameters and its exclusion of gravity have led to the general conviction that it is merely an approximation to a larger and more complete theory. Perturbative calculations play a vital role in both the exploration of, and the search for physics beyond, the Standard Model. This is in-
<table>
<thead>
<tr>
<th>Particle</th>
<th>Mass (GeV)</th>
<th>Coupling Constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>Photon (γ)</td>
<td>0</td>
<td>$\alpha \approx \frac{1}{137}$</td>
</tr>
<tr>
<td>$W^\pm$</td>
<td>80.36 ± 0.12</td>
<td>$G_F = 1.16639 \times 10^{-5}$GeV$^{-2}$</td>
</tr>
<tr>
<td>Z</td>
<td>91.187 ± 0.007</td>
<td>or $\alpha_W(M_W) \approx 0.03$</td>
</tr>
<tr>
<td>gluon (g)</td>
<td>0</td>
<td>$\alpha_s(M_Z) \approx 0.118$</td>
</tr>
</tbody>
</table>

Table 1.1: The gauge bosons of the Standard Model with their masses and couplings.

creasingly true as improving experimental statistics continue to reduce uncertainties, making higher-order corrections more important, and simultaneously making accessible new processes for measurement and comparison with theoretical prediction.

1.1 Overview of the Standard Model

The Standard Model unites the strong and electroweak interactions into a $SU(3)_C \times SU(2)_L \times U(1)$ gauge theory. The various forces are mediated by spin-1 gauge bosons, which are shown together with their masses and coupling constants in Table 1.1 [2].

All matter in the Standard Model is composed of two basic kinds of spin-$\frac{1}{2}$ fermions, quarks and leptons. There are six specific types, or flavors, of each, arranged into three families. The first family contains the electron (e), the electron neutrino ($\nu_e$), and the up (u) and down (d) quarks. The second generation similarly has the muon ($\mu$), the muon neutrino ($\nu_\mu$), and the charm (c) and strange (s) quarks, while the third generation has the tau ($\tau$), the tau neutrino ($\nu_\tau$), and the top (t) and bottom (b) quarks. Each family is identically charged, differing only in the masses of the particles. The quarks all transform as fundamental triplets of the color gauge group $SU(3)_C$ while the leptons are all colorless singlets. The left-handed
<table>
<thead>
<tr>
<th></th>
<th>Particles and Masses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quarks</td>
<td></td>
</tr>
<tr>
<td>$u$</td>
<td>$d$</td>
</tr>
<tr>
<td>$c$</td>
<td>$s$</td>
</tr>
<tr>
<td>$t$</td>
<td>$b$</td>
</tr>
<tr>
<td>Leptons</td>
<td></td>
</tr>
<tr>
<td>$e$</td>
<td>$\nu_e$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$\nu_\mu$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>$\nu_\tau$</td>
</tr>
</tbody>
</table>

Table 1.2: The matter content of the Standard Model, comprising three families of quarks and leptons.

components of both quarks and leptons are paired into fundamental doublets under the weak gauge group $SU(2)_L$, while the right-handed components (absent for the neutrinos) are singlets. The upper member of each quark doublet has electric charge $\pm 2/3$, and the lower member has charge $-1/3$, while for the leptons the corresponding numbers are $-1$ and $0$. Table 1.2 shows the masses of all the matter particles [2, 3, 4].

The final, and as yet undiscovered, particle in the Standard Model is the spin-0 Higgs boson. Both electroweak symmetry breaking and the generation of mass are thought to proceed via the so-called Higgs mechanism. The various particle masses are thus currently accommodated rather than predicted by theory. A consequence of the varied particle masses is the existence of mixing among the quark mass eigenstates to form the weak interaction eigenstates, paramaterized by the unitary $3 \times 3$ Cabibbo-Kobayashi-Maskawa (CKM) matrix. The existence of a nontrivial phase in the CKM matrix is the origin of the phenomena of charge-conjugation-parity (CP) violation.

The Standard Model in its simplest version thus depends on 18 independent parameters: three gauge couplings, eleven fermion and boson masses, three quark mixing angles, and one weak phase. If neutrinos have mass, there may be up to seven more mass and mixing parameters (or more if they are Majorana particles). In addition,
there can be in principle two CP-violating vacuum angles associated with $SU(3)_C$ and $SU(2)_L$. The accurate measurement of these parameters makes up a large part of the current worldwide experimental program in high-energy physics. The explanation for the observed mass hierarchy, as well as the puzzle of generational structure, are the major outstanding unsolved problems in theoretical physics today.

1.2 The Top Quark

At approximately 176 GeV, the top quark is by far the heaviest of the known elementary particles. Because top is so much heavier than the other fermions, it may be special in some sense. At the very least, its mass makes it a unique laboratory for studying the Standard Model as well as a sensitive probe in searching for physics beyond the Standard Model.

There is another reason why top-quark physics is especially interesting. Because the top quark is heavier than the $W$ boson, it can decay freely to a $W$ and a bottom quark, and consequently its lifetime is extremely short, $\tau_t^{-1} \approx (1.5 \text{ GeV})^{-1}$, less than the characteristic time scale of nonperturbative quantum chromodynamics (QCD), $\Lambda_{\text{QCD}}^{-1} \approx (200 \text{ MeV})^{-1}$. This fact makes the phenomenology of the top quark profoundly different from that of the other quarks. In top-quark physics, one is generally not concerned with form factors, decay constants, and other such parameterizations of ill-understood nonperturbative effects. The absence of these effects allows for unprecedented precision studies of perturbative QCD.

Of the four Standard Model parameters that are associated with the top quark —
its mass, $m_t$, and the three CKM matrix elements: $V_{tb}$, $V_{ts}$, and $V_{td}$ — only the first has been directly measured. There are constraints on the others based on indirect measurements and the assumption that there is no fourth generation of matter. A large part of the motivation for Chapter 2 of this work is the potential to measure $V_{tb}$ directly in single-top-quark production [5].

1.3 Perturbation Theory and its Limits

The ability to calculate observable quantities as power series in a small coupling constant is at the very heart of theoretical high-energy physics. And yet the ultimate limits of such expansions are only poorly understood. It has been known for a long time that perturbative expansions in quantum electrodynamics (QED) and quantum chromodynamics (QCD), even after renormalization, are not convergent series [6]. This calls into question the fundamental accuracy of perturbative predictions. General mathematical methods of treating divergent series, notably Borel summation, confirm that all QED and QCD series are plagued by nonperturbative ambiguities, including so-called “infrared renormalons,” that scale as powers of $e^{\frac{-1}{\beta \alpha(Q)}}$ where $\beta$ is the first coefficient of the renormalization-group beta function and $\alpha(Q)$ is the appropriate coupling evaluated at a typical momentum scale $Q$ [7]. These terms are extremely small in QED, but are powers of $\Lambda_{\text{QCD}}/Q$ in QCD, i.e., these ambiguities in QCD are suppressed at high momentum transfer, exactly where the asymptotic freedom of the theory permits a perturbative approach. Nevertheless, such terms may not be negligible in the context of precision studies. In addition, studying the
essential connections between the large-order behavior of perturbative series and non-perturbative contributions to QCD processes should refine our understanding of the theory. It is this idea that motivates Chapters 3 and 4 of this work [8, 9].
Chapter 2

QCD and Yukawa Corrections to $q\bar{q} \to t\bar{b}$

In this chapter I calculate the next-to-leading-order cross section for the weak process $q\bar{q} \to t\bar{b}$, which produces a top quark and a anti-bottom quark via a virtual s-channel $W$ boson (Fig. 2.1) [10]. The most important correction to the $O(\alpha_W^2)$ leading-order cross section is the QCD correction of $O(\alpha_s)$. The Yukawa correction of $O(\alpha_W m_t^2/M_W^2)$, arising from the interactions of Higgs bosons and the scalar components of virtual $W$ and $Z$ vector bosons, can also be important as it dominates the ordinary $O(\alpha_W)$ electroweak correction in the large $m_t$ limit. For the known value of the top-quark pole mass, $m_t = 173.6\pm 5.5$ GeV [4], the Yukawa correction is expected to be at least as large as the ordinary electroweak correction.

![Diagram](image)

Figure 2.1: Single-top-quark production via $qq \to t\bar{b}$. 

7
A precise theoretical calculation of the cross section for $q\bar{q} \rightarrow t\bar{b}$ is valuable for a number of reasons. The cross section obviously determines the yield of single top quarks produced via this process. More importantly, the coupling of the top quark to the $W$ boson in $q\bar{q} \rightarrow t\bar{b}$ is proportional to the Cabibbo-Kobayashi-Maskawa (CKM) matrix element $V_{tb}$, one of the few Standard Model parameters not yet measured experimentally. If there are only three generations, unitarity of the CKM matrix implies that $|V_{tb}|$ must be very close to unity ($0.9989 < |V_{tb}| < 0.9993$) [2]. However, if there is a fourth generation, $|V_{tb}|$ could be anything between (almost) zero and unity, depending on the amount of mixing between the third and fourth generations. Measurement of the $q\bar{q} \rightarrow t\bar{b}$ cross section, coupled with an accurate theoretical calculation, may provide the best direct measurement of $|V_{tb}|$ [10]. Finally, in addition to being interesting in its own right, $q\bar{q} \rightarrow t\bar{b}$ is a significant background to other processes, such as $q\bar{q} \rightarrow WH$ with $H \rightarrow bb$, where $H$ is the Higgs boson [11].

This chapter is organized as follows. In Section 2.1 I compare $q\bar{q} \rightarrow t\bar{b}$ to a similar process called $W$-gluon fusion. In Section 2.2 I present the $O(\alpha_s)$ QCD corrections to both the initial and final states, and discuss their dependence on the renormalization and factorization scales. In Section 2.3 I present the $O(\alpha_W m_t^2/M_{W^*}^2)$ Yukawa correction. In Section 2.4 I present a summary of results and a table of full next-to-leading-order cross sections at the Tevatron and the LHC. Finally, in Section 2.5 I discuss the uncertainty in my results caused by the uncertainty in our knowledge of the top-quark mass.
2.1 Comparison with \( W \)-gluon fusion

In some ways, \( q\bar{q} \to t\bar{b} \) is similar to the \( W \)-gluon fusion process, shown in Fig. 2.2 [12]. However, where that process involves a space-like \( W \) boson with negative mass-squared, i.e., \( q^2 < 0 \), \( q\bar{q} \to t\bar{b} \) proceeds via a time-like \( W \) boson with \( q^2 > (m_t + m_b)^2 \). Thus these two processes, along with the decay of the top quark, \( t \to Wb \) — where the \( W \) boson has \( q^2 \approx M_{W}^2 \) — probe complementary aspects of the third generation’s weak charged current. The kinematic distributions of the final-state particles in the two processes also differ significantly. In \( W \)-gluon fusion, there is an additional jet present, and the \( \bar{b} \) quark is usually produced at low transverse momentum, while in \( q\bar{q} \to t\bar{b} \), the \( \bar{b} \) quark recoils against the \( t \) quark with high transverse momentum.

At the Fermilab Tevatron (\( \sqrt{s} = 2\) TeV \( p\bar{p} \) collider), the sum of the cross sections for \( q\bar{q} \to t\bar{b} \) and \( q\bar{q} \to \bar{t}b \) is roughly a factor of seven smaller than the \( tt \) production cross section [13], and about a factor of two smaller than the \( W \)-gluon-fusion cross section [12]. Nevertheless, a recent study indicates that with double \( b \) tagging, a signal is observable at the Tevatron with 2–3 \( fb^{-1} \) of integrated luminosity [10]. Unfortunately, despite the fact that the \( q\bar{q} \to t\bar{b}, \bar{t}b \) cross section is larger at the CERN
Large Hadron Collider (LHC, $\sqrt{s} = 14$ TeV $pp$ collider), the signal will likely be obscured by backgrounds from the even larger $t\bar{t}$ and $W$-gluon fusion processes, which are initiated by gluons [10].

An important feature of $q\bar{q} \to t\bar{b}$ is the accuracy with which the cross section can be calculated. The top-quark mass sets a natural scale for the process that is much larger than $\Lambda_{QCD}$, so calculations are performed in a regime where perturbative QCD is most reliable. The correction to the initial state is identical to that occurring in the ordinary Drell-Yan process $q\bar{q} \to W^* \to \ell\nu$ (where $W^*$ denotes a virtual $W$ boson), which has been calculated to $O(\alpha_s^2)$ [14]. Furthermore, by experimentally measuring the Drell-Yan cross section, the initial quark-antiquark flux can be independently determined. Since the longitudinal momentum of the neutrino cannot be reconstructed, the $q^2$ of the $W^*$ cannot be determined on an event-by-event basis, and so this yields only a constraint on the quark-antiquark flux, rather than a direct measurement. This provides a check on the parton distribution functions, and allows the reduction of systematic errors. The parton distribution functions are not expected to be a large source of uncertainty, as the dominant contribution to the cross section comes from quark and antiquark distribution functions evaluated at relatively high values of the momentum fraction $x$, where they are well known. There is little sensitivity to the less-well-known gluon distribution function, in contrast to the case of $W$-gluon fusion. The final-state correction to the inclusive cross section is straightforward, and involves no collinear or infrared singularities. The QCD corrections to the initial and final states do not interfere at next-to-leading-order because the $t\bar{b}$ is
in a color singlet if a gluon is emitted from the initial state, but a color octet if it is
emitted from the final state. Interference occurs only at $O(\alpha_s^2)$ with the emission of
two gluons.

2.2 QCD correction

The diagrams which contribute to the $O(\alpha_s)$ correction to $q\bar{q} \rightarrow t\bar{b}$ are shown
in Fig. 2.3. As mentioned in the first section of this chapter, the QCD corrections
to the initial and final states do not interfere at $O(\alpha_s)$. Therefore, I consider the
corrections to the initial and final states separately. To this end, I break up the
process $pp \rightarrow t\bar{b} + X$ into the production of a virtual $W$ boson of mass-squared $q^2$,
followed by its propagation and decay into $t\bar{b}$.

The production cross section of the virtual $W$ boson is formally identical to that
of the Drell-Yan process, to all orders in QCD. The modulus squared of the decay
amplitude of the virtual $W$ boson, integrated over the phase space of all final-state particles, is obtained by the application of Cutkosky's rules as twice the imaginary part of the self-energy of the $W$ boson due to a $t\bar{b}$ loop. Furthermore, because the current to which the $W$ boson couples in the initial state is conserved to all orders in QCD (for massless quarks), it is sufficient to consider only the $-g^{\mu\nu}$ term in the $W$-boson propagator and self-energy. The differential cross section may be written as

$$\frac{d\sigma}{dq^2}(p\bar{p} \rightarrow t\bar{b} + X) = \sigma(p\bar{p} \rightarrow W^* + X) \frac{\text{Im} \Pi(q^2, m_t^2, m_b^2)}{\pi(q^2 - M_W^2)^2} \quad (2.1)$$

where $\Pi$ is the coefficient of the $-g^{\mu\nu}$ term of the self-energy of a $W$ boson with mass-squared $q^2$. The total cross section is obtained by integrating over $q^2$. This equation is valid to $O(\alpha_s)$, but not beyond, because it neglects the interference between the QCD corrections to the initial and final states.

Using this procedure, I can easily obtain the cross section for $p\bar{p} \rightarrow t\bar{b}$. I begin with the well-known leading-order Drell-Yan cross section:

$$\sigma(p\bar{p} \rightarrow W^* + X) = \sum_{i,j} \int \int dx_1 dx_2 [q_i(x_1)\bar{q}_j(x_2) + \bar{q}_i(x_1)q_j(x_2)] |V_{ij}|^2 \quad (2.2)$$

$$\times \frac{\pi^2\alpha_W}{3} \delta(x_1 x_2 S - q^2)$$

where $\alpha_W = g^2/4\pi \equiv \sqrt{2}G_\mu M_W^2/\pi$, $S$ is the square of the total hadronic center-of-mass energy, $q$ and $\bar{q}$ are the parton distribution functions, $\mu_F$ is the factorization scale, and the sum on $i$ and $j$ runs over all contributing quark-antiquark combinations.

At leading order, the coefficient of the $-g^{\mu\nu}$ term in the imaginary part of the $W$-
boson self-energy is

$$\text{Im} \, \Pi(q^2, m_t^2, m_b^2) = \frac{\alpha_W \lambda^{1/2} |V_{tb}|^2}{2} \left[ 1 - \frac{m_t^2 + m_b^2}{2q^2} - \frac{(m_t^2 - m_b^2)^2}{2q^4} \right]$$  \hspace{1cm} (2.3)

where $\lambda$ is the triangle function associated with two-particle phase space,

$$\lambda = \lambda(q^2, m_t^2, m_b^2) = q^4 + m_t^4 + m_b^4 - 2q^2 m_t^2 - 2q^2 m_b^2 - 2m_t^2 m_b^2.$$  \hspace{1cm} (2.4)

From Eq. (2.1), the differential cross section is thus

$$\frac{d\sigma}{dq^2}(p\bar{p} \rightarrow t\bar{b} + X) = \sum_{i,j} \int \int dx_1 dx_2 \, |q_i(x_1)\bar{q}_j(x_2) + \bar{q}_i(x_1)q_j(x_2)|^2 \, |V_{ij}|^2 \times \frac{\pi \alpha_W^{1/2} |V_{tb}|^2}{12(q^2 - M_W^2)^2} \left[ 1 - \frac{m_t^2 + m_b^2}{2q^2} - \frac{(m_t^2 - m_b^2)^2}{2q^4} \right] \delta(x_1 x_2 S - q^2).$$  \hspace{1cm} (2.5)

At leading order, the integration over $q^2$ to obtain the total cross section is trivial due to the delta function. To calculate at next-to-leading order, however, it is necessary to perform the integration numerically.

The $O(\alpha_s)$ corrections to the Drell-Yan process [15] and the $W$-boson self energy [16] were both calculated many years ago. I use the expression for $\sigma(p\bar{p} \rightarrow W^* + X)$ as given in Eqs. (9.5) and (12.3) of [17], and $\text{Im} \, \Pi$ as derived from Eq. (3.3) of the second paper of [16]. I use $m_t = 175$ GeV, $m_b = 5$ GeV, $M_W = 80.33$ GeV, $|V_{tb}| = 1$, $G_F = 1.16639 \times 10^{-5} \text{ GeV}^{-2}$, and $\alpha_s$ as given by the parton distribution functions.

The calculation of the initial-state correction includes divergences arising from collinear parton emission. These divergences cancel with corresponding divergences present in the QCD correction to the parton distribution functions. The finite terms remaining depend on the factorization scale $\mu_F$, both through the parton distribution

---

\[^1\]The exact correspondence between my notation and theirs is $\text{Im} \, \Pi = 3\pi\alpha_W |V_{tb}|^2 \text{Im}(\Pi_T^3 + \Pi^4_T)$. 

13
functions and explicitly in the partonic cross section. The variation of the leading-order and next-to-leading-order cross sections with $\mu_F/\sqrt{q^2}$, where $\sqrt{q^2}$ is the mass of the virtual $W$ boson,\textsuperscript{2} is shown in Fig. 2.4 at both the Tevatron and the LHC. The leading-order cross section is calculated with the CTEQ3L leading-order parton distribution functions, and the next-to-leading-order cross section with the CTEQ3M next-to-leading-order parton distribution functions \cite{18}. The leading-order cross section varies considerably with $\mu_F$, while the next-to-leading-order cross section is appreciably less sensitive. The next-to-leading-order cross section shown in Fig. 4 contains only the initial-state correction. For $\mu_F = \sqrt{q^2}$ the initial-state correction is $+38\%$ at the Tevatron and $+33\%$ at the LHC.\textsuperscript{3} In what follows, I set $\mu_F = \sqrt{q^2}$.

\textsuperscript{2}I have chosen to refer the scale $\mu_F$ to the $q^2$ of the virtual $W$ boson because this is the quantity which appears in the factorization logarithms. The factorization scale $\mu_F$ varies when integrating over $q^2$ to obtain the total cross section.

\textsuperscript{3}If both the leading-order and next-to-leading-order cross sections are calculated with the CTEQ3M next-to-leading-order parton distribution functions, the initial-state correction is $+27\%$ at the Tevatron and $+15\%$ at the LHC. Thus $+9\%$ of the initial-state correction at the Tevatron, and $+18\%$ at the LHC, is due to the increase in the leading-order cross section when it is calculated with next-to-leading-order parton distribution functions.
Figure 2.5: Renormalization-scale dependence of the leading-order (LO) and next-to-leading-order (NLO) cross sections for $q\bar{q} \rightarrow t\bar{b}, \bar{t}b$ at the Tevatron and the LHC.

The cross section at next-to-leading order also depends on the renormalization scale, $\mu_R$, at which $\alpha_s$ is evaluated. In Fig. 2.5 we show the next-to-leading-order cross section, including both initial- and final-state corrections, as a function of $\mu_R/\sqrt{q^2}$, at both the Tevatron and the LHC. The dependence of the cross section on the renormalization scale first appears at next-to-leading order and is therefore mild. In what follows, I set $\mu_R = \sqrt{q^2}$. The final-state correction increases the cross section by +18% of the leading-order cross section at the Tevatron and +17% at the LHC.

In Fig. 2.6 I show the leading-order and next-to-leading-order differential cross section as a function of the mass of the virtual $W$ boson, $\sqrt{q^2}$, at both the Tevatron and the LHC. Also shown are the separate $O(\alpha_s)$ corrections from the initial and final states. These corrections have different shapes from the leading-order cross section, and from each other. In order to observe $t\bar{b}$ production experimentally, it is necessary to detect the $b$ quark [10]. Thus the measured cross section will exclude some region
Figure 2.6: Differential cross section for $q \bar{q} \rightarrow t \bar{b}, tb$ versus the mass of the virtual $s$-channel $W$ boson, at the Tevatron and the LHC. Both the leading-order (LO) and next-to-leading order (NLO) cross sections are shown, as well as the separate contributions from the initial-state (IS) and final-state (FS) corrections.

near threshold, where the $\bar{b}$ quark does not have sufficient transverse momentum to be detected with high efficiency. Therefore the measured cross section, as well as the QCD correction, will depend on the acceptance for the $\bar{b}$ quark.

The final-state correction (Fig. 2.3(d)) includes a diagram where a virtual gluon is exchanged between the outgoing $t$ and $\bar{b}$ quarks. This leads to a Coulomb singularity at the $t\bar{b}$ threshold, associated with the (QCD) Coulomb attraction of the quark and antiquark. Near threshold, $q^2 \rightarrow (m_t + m_b)^2$, the $O(\alpha_s)$ correction to the squared amplitude diverges like $(q^2 - (m_t + m_b)^2)^{-1/2}$. However, this is compensated by the phase-space factor $\lambda^{1/2}$ (see Eq. (2.6)), which vanishes at threshold like $(q^2 - (m_t + m_b)^2)^{1/2}$. The result is that the next-to-leading-order partonic cross section is finite at threshold, with a value

$$\hat{\sigma}_{\text{thresh}} = \frac{\alpha_s^2 \alpha_s 2\pi^2 m_t^2 m_b^2}{3(m_t + m_b)^2[(m_t + m_b)^2 - M_W^2]}. \quad (2.6)$$
The contribution of the threshold region to the hadronic cross section is negligible, as can be seen from Fig. 2.6.

If the top and bottom quarks were stable, they would form quarkonium bound states just below threshold [19]. One can estimate the distance below threshold that the ground state would occur, by analogy with the hydrogen atom, to be \( E \approx (C_F \alpha_s)^2 m_t^2 / 2 \approx 50 \text{ MeV} \) where \( m_b \) is the approximate reduced mass of the system, and \( C_F = 4/3 \) is the usual SU(3) group theory factor associated with the fundamental representation. The formation time of the ground state is approximately \( 1/E \). This is much greater than the top-quark lifetime, \( \Gamma_t^{-1} \approx (1.5 \text{ GeV})^{-1} \), so there is not sufficient time for quarkonium bound states to form [20].

Because the top-quark width is small compared to its mass, interference between the corrections to production and decay amplitudes has a negligible effect, of order \( \alpha_s \Gamma_t / m_t \), on the total cross section [21]. This interference also has a negligible effect on differential cross sections, such as the invariant-mass distribution of the decay products of the top quark [22].

2.3 Yukawa correction

The diagrams which contribute to the \( O(\alpha_W m_t^2 / m_W^2) \) Yukawa correction to \( q\bar{q} \rightarrow t\bar{t}b \) are shown in Fig. 2.7. The dashed lines represent the Higgs boson and the unphysical scalar \( W \) and \( Z \) bosons associated with the Higgs field in the \( R_\xi \) gauge. The effect of a top-quark loop in the \( W \)-boson propagator, which might be expected to contribute a term of Yukawa strength, is absorbed by the renormalized weak cou-
Figure 2.7: $\mathcal{O}(\alpha w m_t^2/M_W^2)$ corrections to $q\bar{q} \to t\bar{b}$. The dashed lines represent the Higgs boson and the unphysical scalar $W$ and $Z$ bosons in $\mathcal{R}_\xi$ gauge: (a) wavefunction renormalization, (b) vertex correction.

pling constant, which I express in terms of $G_\mu$, the Fermi constant measured in muon decay ($\alpha w = g^2/4\pi = \sqrt{2}G_\mu M_W^2/\pi$). Standard Feynman integral techniques with dimensional regularization were used to calculate the loop diagrams [24] in the approximation where the bottom quark is massless. Other parameters are $m_t = 175$ GeV, $M_W = 80.33$ GeV, $|V_{tb}| = 1$, and $G_\mu = 1.16639 \times 10^{-5}$ GeV$^{-2}$.

In the $m_\phi = 0$ limit, the matrix element of the $t\bar{b}$ weak current may be written

$$i\bar{u}(p_t)\Gamma^{\mu A}u(p_b) = \left(\frac{-igT^A}{2\sqrt{2}}\right) \left\{ \bar{u}(p_t)\gamma^\mu(1 - \gamma^5)u(p_b) + \right. $$

$$\left. \left(\frac{m_t^2 G_\mu}{8\sqrt{2}\pi^2}\right) \bar{u}(p_t) \left[ \gamma^\nu F_1(q^2) + \frac{(p_t^\nu - p_b^\nu)}{m_t} F_2(q^2) \right] (1 - \gamma^5)u(p_b) \right\}$$

(2.7)

where $T^A$ is an SU(3) matrix [$\text{Tr}(T^A T^B) = \frac{1}{2}\delta^{AB}$]; $p_t$ and $p_b$ are the outgoing four-momenta of the $t$ and $\bar{b}$ quarks, respectively; and the form factors $F_1$ and $F_2$ are functions of $q^2 = (p_t + p_b)^2$ and the Higgs-boson mass.

$$F_1 = \frac{1}{2}\left[4C_{24}(\xi M_W^2, M_H^2, m_t^2; q^2, m_t^2, 0) + 4C_{24}(\xi M_Z^2, \xi M_W^2, m_t^2; q^2, m_t^2, 0) \right. $$

$$+ B_1(M_H^2, m_t^2; m_t^2) + (M_H^2 - 4m_t^2) B'_0(M_H^2, m_t^2; m_t^2) $$

$$+ B_1(\xi M_Z^2, m_t^2; m_t^2) + \xi M_Z^2 B'_0(\xi M_Z^2, m_t^2; m_t^2) $$

$$+ B_1(\xi M_W^2, 0; m_t^2) + (\xi M_W^2 - m_t^2) B'_0(\xi M_W^2, 0; m_t^2) $$

$$\left. + B_1(M_Z^2, m_t^2; m_t^2) + B'_0(M_Z^2, m_t^2; m_t^2) \right]$$
\[ P_2 = m_t^2 \{ C_{23}(\xi M_W^2, M_H^2, m_t^2; q^2, m_t^2, 0) + C_{24}(\xi M_H^2, m_t^2, q^2, m_t^2, 0) \}
\]
\[ + C_{21}(\xi M_W^2, M_H^2, m_t^2; q^2, m_t^2, 0) + C_{21}(\xi M_H^2, m_t^2, q^2, m_t^2, 0) \]
\[ + 2C_{21}(\xi M_W^2, M_H^2, m_t^2; q^2, m_t^2, 0) \]

The masses-squared of the unphysical scalar bosons are \( \xi M_W^2, \xi M_H^2 \). In the numerical calculations, I set \( \xi = 0 \) (Landau gauge). The integrals were reduced to the standard one, two, and three point scalar loop integrals and then evaluated with the aid of the code FF [23]. The notation is adopted from [24]; the arguments of the functions give the internal masses-squared followed by the external momenta squared.

The fractional change in the differential cross section as a function of the \( q^2 \) of the virtual \( W \) boson is

\[
\frac{\Delta d\sigma_Y / d\sqrt{q^2}}{d\sigma_{LO} / d\sqrt{q^2}} = \left( \frac{m_t^2 G_F}{8\sqrt{2}\pi} \right) \left[ 2F_1(q^2) + F_2(q^2) \frac{(q^2 - m_t^2)^2}{q^2 - \frac{1}{2} q^2 m_t^2 - \frac{1}{2} m_t^4} \right].
\] (2.8)

The fractional change in the total cross section, \( \Delta\sigma_Y / \sigma_{LO} \), is plotted in Fig. 2.8 vs. the Higgs-boson mass, \( M_H \), at both the Tevatron and the LHC. For values of \( M_H \) between 60 GeV and 1 TeV, the absolute value of the Yukawa correction is never more than one percent of the leading-order cross section. Thus the Yukawa correction is negligible for this process. This was also found to be the case for \( t\bar{t} \) production [25]. As previously mentioned, the ordinary weak correction is expected to be comparable to the Yukawa correction, so it too should be negligible. The complete calculation of the ordinary weak correction would require a set of parton distribution functions with weak corrections included. Such a set is not available at this time. The Yukawa
Figure 2.8: Fractional change in the total cross section for $qar{q} \rightarrow t\bar{b}, \bar{t}b$ due to the Yukawa correction vs. the Higgs-boson mass at the Tevatron and the LHC.

correction could potentially be much larger in models with enhanced couplings of Higgs bosons to top or bottom quarks.

2.4 Full Next-to-leading-order Cross Section

The cross section for $q\bar{q} \rightarrow t\bar{b}, \bar{t}b$ at both the Tevatron and the LHC is given in Table 2.1. The leading-order cross section, next-to-leading-order cross section including only the initial-state QCD correction, and the full next-to-leading order cross section are given. The factorization and renormalization scales are both set equal to $\sqrt{q^2}$, the mass of the virtual W boson. I give results for three different sets of next-to-leading-order parton distribution functions: CTEQ3M [18], MRS(A'), and MRS(G) [26]. The QCD correction to the cross section is quite significant: about $+54\%$ at the Tevatron and $+50\%$ at the LHC, with the leading-order cross section evaluated with leading-order parton distribution functions, and the next-to-

\footnote{The leading-order CTEQ cross section is calculated with the CTEQ3L leading-order parton distribution functions.}
leading-order cross section evaluated with next-to-leading-order parton distribution functions. The size of the $\mathcal{O}(\alpha_s)$ correction improves the outlook for observation of this process in Run II at the Tevatron.

As shown in Fig. 2.4, varying the factorization scale between one half and twice $\sqrt{q^2}$ changes the cross section by only $\pm 2\%$. Varying the renormalization scale over this same range yields a similar change in the cross section, as shown in Fig. 2.5. Using these results to estimate the contribution from higher-order QCD corrections, I conclude that the uncertainty in the cross section is at the level of $\pm 4\%$. This conclusion is supported by the known next-to-next-to-leading-order correction to the Drell-Yan process, which is about $2\%$ (in the modified minimal subtraction (MS) scheme) [14].

It is difficult to reliably ascertain the uncertainty in the cross section from the parton distribution functions at this time. The small difference in the next-to-leading-order cross sections using MRS($A'$) and MRS(G) supports the contention that the calculation is insensitive to the gluon distribution function. The difference between the cross section using CTEQ3M and MRS($A'$) suggests that the uncertainty in the cross section from the parton distribution functions is on the order of $\pm 2\%$. However, since each set of parton distribution functions represents the best fit to some set of data, the uncertainty is certainly larger than this. Therefore, I somewhat arbitrarily assign an uncertainty of $\pm 4\%$ from the parton distribution functions.

For my final estimate of the cross section, I average the next-to-leading-order cross

---

5 If next-to-leading-order parton distribution functions are used at both leading and next-to-leading order, the correction is about $+45\%$ at the Tevatron and $+32\%$ at the LHC.
<table>
<thead>
<tr>
<th>$m_t = 175$ GeV, $\mu_R = \mu_F = \sqrt{Q^2}$</th>
<th>CTEQ3L.3M</th>
<th>MRS(A')</th>
<th>MRS(G)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tevatron $\sqrt{S} = 2$ TeV</td>
<td>$\sigma_{LO}$</td>
<td>.578</td>
<td>.601</td>
</tr>
<tr>
<td>$\sigma_{NLO(IS)}$</td>
<td>.789</td>
<td>.766</td>
<td>.758</td>
</tr>
<tr>
<td>$\sigma_{NLO}$</td>
<td>.894</td>
<td>.868</td>
<td>.860</td>
</tr>
<tr>
<td>$\sigma_{NLO}$ (avg)</td>
<td>.88 ± .05 pb</td>
<td></td>
<td></td>
</tr>
<tr>
<td>LHC $\sqrt{S} = 14$ TeV</td>
<td>$\sigma_{LO}$</td>
<td>6.76</td>
<td>7.83</td>
</tr>
<tr>
<td>$\sigma_{NLO(IS)}$</td>
<td>9.02</td>
<td>9.01</td>
<td>9.03</td>
</tr>
<tr>
<td>$\sigma_{NLO}$</td>
<td>10.19</td>
<td>10.17</td>
<td>10.21</td>
</tr>
<tr>
<td>$\sigma_{NLO}$ (avg)</td>
<td>10.2 ± 0.6 pb</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2.1: Leading-order (LO) and next-to-leading-order (NLO) cross sections (pb) for $qq \rightarrow t\bar{b}, \bar{t}b$ at the Tevatron and the LHC for three different sets of parton distribution functions (PDFs). The NLO cross section including only the initial state (IS) correction is also given.

sections using the CTEQ3M and MRS(A') parton distribution functions and assign an uncertainty of ±6%, which reflects the uncertainties above, added in quadrature. The final result for the cross section for $qq \rightarrow t\bar{b}, \bar{t}b$ is then .88 ± .05 pb at the Tevatron, and 10.2 ± 0.6 pb at the LHC.

### 2.5 Mass Dependence

Up to this point, I have neglected the uncertainty in my calculation due to the uncertainty in the top-quark mass. A plot of the next-to-leading order cross section for $qq \rightarrow t\bar{b}, \bar{t}b$ as a function of the top-quark mass is shown in Fig. 2.9. The current uncertainty in the top-quark mass is ±5.5 GeV [4], based on data from Run I at the Tevatron. This yields an uncertainty of ±14% in the cross section at the Tevatron. The uncertainty in the mass is expected to decrease to ±3 GeV in Run II [27], corresponding to an uncertainty in the cross section of ±8%. A high-luminosity Tevatron might be capable of reducing the uncertainty in the mass to ±1 GeV [28], for an uncertainty in the cross section of ±3%. The uncertainty in the cross section at
Figure 2.9: Next-to-leading order cross section for $q\bar{q} \rightarrow t\bar{b}, t\bar{b}$ as a function of the top-quark mass.

the LHC is expected to be limited only by systematics, so a measurement of the top mass to less than 1 GeV, and a corresponding prediction of the cross section accurate to 1%, is certainly conceivable.
Chapter 3

Top-Quark Pole Mass

The mass of the top quark has been measured with impressive accuracy, \( m_t = 175.6 \pm 5.5 \) GeV [4], by the CDF and D0 experiments at the Fermilab Tevatron. The uncertainty will be reduced even further, to perhaps 1-2 GeV, with additional running at the Tevatron [27], or at the CERN Large Hadron Collider [29]. High-energy e^+e^- [30] or \( \mu^+\mu^- \) [31] colliders operating at the \( t\bar{t} \) threshold hold the promise of yet more precise measurements of \( m_t \), to 200 MeV or even better.

With such increasingly-precise measurements on the horizon, it is important to have a firm grasp of exactly what is meant by the top-quark mass. Thus far the top-quark mass has been experimentally defined by the position of the peak in the invariant-mass distribution of the top-quark's decay products, a W boson and a b-quark jet [4]. This closely corresponds to the pole mass of the top quark, defined as the real part of the pole in the perturbative top-quark propagator. The perturbative propagator of a top quark with four-momentum \( p \) has a pole at the complex position \( \sqrt{p^2} = m_{\text{pole}} - i\Gamma \), and yields a peak in the \( Wb \) invariant-mass distribution (for experimentally-accessible real values of \( p \)) when \( \sqrt{p^2} \approx m_{\text{pole}} \).
The pole mass of a stable quark is well-defined in the context of finite-order perturbation theory [32]. However, the all-orders resummation of a certain class of diagrams, associated with “infrared renormalons”, indicates that the pole mass of a stable heavy\(^1\) quark is ambiguous by an amount proportional to \(\Lambda_{\text{QCD}}\), as a result of nonperturbative QCD [33, 34]. Physically, this is a satisfying result, because I believe that quarks are permanently confined within hadrons, precluding the unambiguous definition of a quark pole mass [35].

The top quark decays very quickly, having a width \(\Gamma \approx 1.5\) GeV, approximately an order of magnitude greater than the strong-interaction energy scale \(\Lambda_{\text{QCD}} \approx 200\) MeV. Such a short lifetime means that the top quark decays before it has time to hadronize [36, 37]. The large top-quark width can act as an infrared cutoff in the calculation of physical quantities, insulating the top quark from the effects of nonperturbative QCD [37, 20, 38, 39].

Motivated by these facts, one may ask whether the top-quark pole mass is free of the ambiguities associated with nonperturbative QCD. The purpose of this chapter is to demonstrate that this is not the case. The top-quark pole mass, like the mass of a stable heavy quark, is unavoidably ambiguous by an amount proportional to \(\Lambda_{\text{QCD}}\). I first give a general argument for the existence of such an ambiguity. I then give a heuristic argument that the ambiguity is proportional to \(\Lambda_{\text{QCD}}\), using the specific example of the \(Wb\) invariant-mass distribution. Finally, I derive an analytic expression for the ambiguity using infrared renormalon techniques.

\(^1\)Heavy here means \(m \gg \Lambda_{\text{QCD}}\).
3.1 General Argument

Consider a scattering process with asymptotic states consisting of stable particles. One may ask if it is possible for the scattering amplitude to have a pole at the mass of a stable quark. This would correspond to a quark propagator connecting two sub-amplitudes, as depicted in Fig. 3.1; the pole in the quark propagator would correspond to the pole in the amplitude. Such a configuration is impossible, however, because the sub-amplitudes which the quark propagator connects have external states which are color singlets (due to confinement), while the quark is a color triplet, so color is not conserved. Thus there cannot be a pole in the amplitude at the quark mass.

This argument applies equally well to an unstable quark, such as the top quark. The fact that the quark is unstable plays no role in the argument; the width only shifts the imagined pole in the propagator into the complex plane. As in the case of a stable quark, there cannot be a pole in the amplitude, regardless of how short-lived the quark. In particular, the fact that the top-quark lifetime is much less than $\Lambda_{\text{QCD}}^{-1}$ is irrelevant.

Such an argument implies that the nonperturbative aspect of the strong interac-
Figure 3.2: The production and decay of a top quark in (a) perturbation theory, and (b) nonperturbatively.

In perturbation theory, the final state is a $W$ and a $b$ quark, as depicted in Fig. 3.2(a). However, the $b$ quark manifests itself experimentally as a jet of colorless hadrons, due to confinement. At least one of the quarks which resides in these hadrons comes from elsewhere in the diagram, and cannot be considered as a decay product of the top quark, as depicted in Fig. 3.2(b). This leads to an irreducible uncertainty in the $Wb$ invariant mass of $O(\Lambda_{\text{QCD}})$, and hence an ambiguity of this amount in the extracted top-quark pole mass.

3.2 Infrared Renormalons

I turn now to an investigation of the top-quark pole mass from the perspective of infrared renormalons. I first review the argument which demonstrates the existence of a renormalon ambiguity in the pole mass of a stable heavy quark [33, 34]. I then extend the argument to take into account the finite width of the top quark. Finally,
\[ \Sigma^{(1)} = \begin{array}{c}
\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ <!-- By short-distance mass I mean a scale-dependent renormalized mass (such as the \( \overline{\text{MS}} \) mass) evaluated at a scale \( \mu \gg \Lambda_{QCD} \). -->
energy diagram, as shown in Fig. 3.3(a'). One can express this as

$$
\Sigma^{(1)}(m_R, \alpha) = \frac{16 m_R}{3 \beta_0} \sum_{n=0}^{\infty} c_n a^{n+1}
$$

(3.3)

where

$$
a = \frac{\beta_0 \alpha_s(m_R)}{4 \pi}
$$

(3.4)

and $\beta_0$ is the one-loop QCD beta-function coefficient, $\beta_0 = 11 - (2/3)N_f$. Formally, these are the dominant QCD corrections in the “large-$\beta_0$” limit. Thus $\Sigma^{(1)}(m_R, \alpha)$ in Eq. (3.3) is calculated at leading order in $\alpha_s$, but to all orders in $a$.

For large $n$ the coefficients $c_n$ grow factorially, and are given by [33, 34, 40]

$$
c_n \xrightarrow{n \to \infty} e^{-C/2} 2^n n!
$$

(3.5)

where $C$ is a finite renormalization-scheme-dependent constant. In the $\overline{\text{MS}}$ scheme, $C = -5/3$. The series in Eq. (3.3) is therefore divergent. One can attempt to sum the series using the technique of Borel resummation [7]. The Borel transform (with respect to $a$) of the self-energy is obtained from the series coefficients, Eq. (3.5), via

$$
\tilde{\Sigma}^{(1)}(m_R, u) = \frac{16}{3 \beta_0} m_R \sum_{n=0}^{\infty} \frac{c_n}{n!} u^n
$$

(3.6)

where $u$ is the Borel parameter. Because the coefficients $c_n$ are divided by $n!$ in the above expression, the series has a finite radius of convergence in $u$, and can be analytically continued into the entire $u$ plane. The self-energy is then reconstructed via the inverse Borel transform, given formally by

$$
\Sigma^{(1)}(m_R, \alpha) = \int_{0}^{\infty} du \, e^{-u/a} \tilde{\Sigma}^{(1)}(m_R, u)
$$

(3.7)
The integral in Eq. (3.7) is only formal, because the Borel transform of the quark self-energy possesses singularities on the real $u$-axis, which impede the evaluation of the integral. These singularities are referred to as infrared renormalons because they arise from the region of soft gluon momentum in Fig. 3.3(a'). The series for the self-energy in Eq. (3.3) is therefore not Borel summable.

The divergence of the series for the self-energy is governed by the infrared renormalon closest to the origin, which lies at $u = 1/2$. This renormalon is not associated with the condensate of a local operator, so it cannot be absorbed into a nonperturbative redefinition of the pole mass [33, 34]. Instead, one can choose some ad hoc prescription to circumvent the singularity in the integral. The difference between various prescriptions is a measure of the ambiguity in the pole mass. Estimating the ambiguity as half the difference between deforming the integration contour above and below the singularity gives [34]

$$
\delta m_{\text{pole}} \sim \frac{8\pi}{3\beta_0} e^{-c/2 \Lambda_{\text{QCD}}} \tag{3.8}
$$

so the pole mass is ambiguous by an amount proportional to $\Lambda_{\text{QCD}}$.

### 3.3 Renormalons and the Top-Quark Width

I now include the $O(\alpha_W)$ contribution to the top-quark self-energy shown in Fig. 3.3(b). The pole position is still given by Eq. (3.2), but where $\Sigma^{(1)}(m_R)$ includes both Figs. 3.3(a) and (b). Since Fig. 3.3(b) has an imaginary part, the pole moves off the real axis. The imaginary part of the one-loop pole position defines the tree-level top-quark width via $\text{Im} \Sigma^{(1)}(m_R) \equiv -\frac{i}{2} \Gamma_{\text{tree}}$. As before, to extend the
calculation to all orders in \( \alpha \), I replace Fig. 3.3(a) by Fig. 3.3(a'). This contribution to the pole mass remains the same as for a stable quark, and has the same renormalon ambiguity. Thus, at leading order in \( \alpha_W \), the infrared renormalons do not know about the top-quark width.

The \( O(\alpha_s) \) contribution to the top-quark self-energy learns about the top-quark width if one works to all orders in \( \alpha_W \), via a Schwinger-Dyson representation [41], as shown in Fig. 3.4. The circles on the internal propagators and the vertex in Figs. 3.4(a) and (b) represent the weak corrections to all orders in \( \alpha_W \). The circles in Fig. 3.4(b) also contain one power of \( \alpha_s \). I wish to solve for the pole position as given by the first equality in Eq. (3.2). I denote the pole position at zeroth order in \( \alpha_s \), but to all orders in \( \alpha_W \), by the complex value \( M \), with \( \text{Im} M = -\frac{1}{2} \Gamma \), where \( \Gamma \) is the top-quark width to all orders in \( \alpha_W \). At leading order in \( \alpha_s \), the pole position is then given by

\[
\hat{p}_{\text{pole}} = m_R + \Sigma(M)
\]  

(3.9)

where \( \Sigma(M) \) is given by Figs. 3.4(a) and (b). Again, I extend this calculation to all orders in \( \alpha \) by making \( n \) vacuum-polarization insertions in the gluon propagator, as depicted in Fig. 3.4(a'). This yields a series in \( \alpha \), which I denote by \( \Sigma(M, \alpha) \) in analogy with Eq. (3.3). To investigate whether the width might cut off the infrared renormalons generated by these diagrams, I need only consider the contribution of soft gluons. In the limit of vanishing gluon momentum, the internal quark propagator reduces to \( Z/(\hat{p} - M) \), where \( Z \) is the wavefunction-renormalization factor. The Ward identity tells us that, in this same limit, the dressed vertex is simply \( Z^{-1} \). Thus, in
Figure 3.4: Diagrams contributing to the top-quark self-energy at leading order in $\alpha_s$, but to all orders in eW. (a') replaces (a) when summing to all orders in $\beta_0\alpha_s$.

In the infrared limit, $\Sigma(M, \alpha)$ is formally identical to $\Sigma^{(1)}(m_R, \alpha)$ with $m_R$ replaced by $M$ everywhere. The infrared renormalons, which are associated with the Borel transform with respect to $\alpha$, are unaffected. The width does not act as a cutoff for infrared renormalons, despite the fact that it is much greater than $\Lambda_{QCD}$. I conclude that the ambiguity in the pole mass of the top quark is given by Eq. (3.8), just as for a stable quark.

I next ask whether the top-quark width itself suffers from a similar renormalon ambiguity. Because the first-order calculation yields the top-quark width at tree level only, it is insufficient to address this question. Solving Eq. (3.2) at $O(\alpha_W \alpha_s)$ gives

$$
pole_{\text{pole}} = m_R + \Sigma(m_R + \Sigma(m_R)) = m_R + \Sigma^{(1)}(m_R) + \Sigma^{(2)}(m_R) + \Sigma^{(1)}(m_R) \Sigma^{(1)}(m_R)$$

where the superscripts on $\Sigma$ indicate the order at which it is to be evaluated. The imaginary part of this equation (times $-1/2$) defines the top-quark width at $O(\alpha_W \alpha_s)$.

One may calculate the imaginary part of Eq. (3.10) using the Cutkosky rules.
This reduces to the calculation of the QCD correction to the process $t \to Wh$. The term involving $\Sigma^{(1)}(m_B)$ corresponds to the wavefunction renormalization of the top quark. The presence of renormalons in this process was investigated in [33, 42]. If the width is expressed in terms of the pole mass, then it has an infrared renormalon at $u = 1/2$, corresponding to an ambiguity proportional to $\Lambda_{\mathrm{QCD}}$. However, if the width is expressed in terms of a short-distance mass, such as the $\overline{\mathrm{MS}}$ mass, there is no renormalon at $u = 1/2$, and hence no ambiguity proportional to $\Lambda_{\mathrm{QCD}}$.

### 3.4 Relation Between the Pole Mass and the $\overline{\mathrm{MS}}$ Mass

The ambiguity in the pole mass does not limit the accuracy with which a short-distance mass, such as the $\overline{\mathrm{MS}}$ mass, can be measured. It is sensible to adopt the $\overline{\mathrm{MS}}$ mass as the standard definition of the top-quark mass, as is the convention for the lighter quarks [2]. The relation between the top-quark pole mass and the $\overline{\mathrm{MS}}$ mass evaluated at the pole mass, $m(m_{\mathrm{pole}})$, is known to two loops [43]:

$$m_{\mathrm{pole}} = \overline{m}(m_{\mathrm{pole}}) \left[ 1 + \frac{4 \, \overline{\alpha_s}(m_{\mathrm{pole}})}{3 \, \pi} + 10.95 \left( \frac{\overline{\alpha_s}(m_{\mathrm{pole}})}{\pi} \right)^2 + \cdots \right] + O(\Lambda_{\mathrm{QCD}}) \quad (3.11)$$

where the last term reminds us that the pole mass has an unavoidable ambiguity of $O(\Lambda_{\mathrm{QCD}})$. Given that the pole mass is ambiguous, a better standard would be the $\overline{\mathrm{MS}}$ mass evaluated at the $\overline{\mathrm{MS}}$ mass, which is related to the pole mass by

$$m_{\mathrm{pole}} = \overline{m}(\overline{m}) \left[ 1 + \frac{4 \, \overline{\alpha_s}(\overline{m})}{3 \, \pi} + 8.28 \left( \frac{\overline{\alpha_s}(\overline{m})}{\pi} \right)^2 + \cdots \right] + O(\Lambda_{\mathrm{QCD}}). \quad (3.12)$$

The difference in the coefficients of the two $\overline{\alpha_s}^2$ terms above is exactly $8/3$. 

33
For a top-quark pole mass of $175.6 \pm 5.5$ GeV [4], $m(m) = 166.5 \pm 5.5$ GeV while
\[ \overline{m}(m_{pole}) = 166.0 \pm 5.5 \text{ GeV}. \]
Chapter 4

Nonperturbative Effects in Quarkonia

The states of quarkonia, which have a size much less than the characteristic scale of nonperturbative QCD, $\Lambda_{QCD}^{-1}$, can be accurately described by QCD perturbation theory [44]. However, the quarkonium wave function is nonzero at long distances, of order $\Lambda_{QCD}^{-1}$ and greater, so the states of quarkonia are also influenced by nonperturbative physics. In this chapter I show that perturbation theory encodes this nonperturbative effect in its large-order behavior.

Nonperturbative effects in quarkonia have previously been paramaterized in terms of vacuum condensates. At short distances, the leading such effect is due to the gluon condensate [45]. I argue that this effect is separate from the nonperturbative effect I investigate, which is due to the non-vanishing of the wave function at long distances.

4.1 A Renormalon-Inspired Look at Quarkonia

That perturbation theory is aware of (some) nonperturbative effects via its large-order behavior is familiar from the study of infrared renormalons [7]. Renormalons
correspond to factorial growth of the coefficients of the perturbation series, and are associated with nonperturbative contributions of order $(\Lambda_{\text{QCD}}/Q)^p$, where $Q$ is a characteristic momentum and $p$ is a positive integer. The analysis of this chapter is in the spirit of infrared renormalons, but I find perturbative coefficients that grow less rapidly than factorially, and that are associated with a nonperturbative contribution of order $e^{-Q/\Lambda_{\text{QCD}}}$, rather than a power of $(\Lambda_{\text{QCD}}/Q)$.

At short distances, quarkonia can be described by the QCD analogue of the Coulomb potential \[44\],
\[
V(r) = \frac{4}{3} \frac{\alpha_s(1/r)}{r} \tag{4.1}
\]
with a coupling that depends on the distance between the quark and the antiquark. At distances of order $\Lambda_{\text{QCD}}^{-1}$ and greater, the coupling is large, and Eq. (4.1) is invalid.

To study the effects of the running coupling perturbatively, I expand the evolution equation for the coupling as an infinite sum of terms which depend on the coupling at an arbitrary fixed scale $\mu$, $\alpha_s \equiv \alpha_s(\mu)$:
\[
\alpha_s(1/r) = \frac{\alpha_s}{1 - \beta \alpha_s \ln \mu r} = \alpha_s \sum_{k=0}^{\infty} \left( \beta \alpha_s \ln \mu r \right)^k \tag{4.2}
\]
where
\[
\beta \equiv \frac{1}{2\pi} \left( 11 - \frac{2}{3} N_f \right) \tag{4.3}
\]
is the one-loop beta function for $N_f$ flavors of light quarks. Inserting Eq. (4.2) into Eq. (4.1) gives
\[
V(r) = -\frac{4}{3} \frac{\alpha_s}{r} + \sum_{k=1}^{\infty} V_k(r) \tag{4.4}
\]
with

$$V_k(r) = -\frac{4}{3} \frac{\alpha_s}{r} (\beta \alpha_s \ln \mu r)^k.$$  \hspace{1cm} (4.5)

Diagrammatically, the potential in Eq. (4.4) corresponds to the sum of vacuum-polarization insertions in the tree-level potential, as shown in Fig. 4.1.

The first term in Eq. (4.4) is the QCD Coulomb potential with a fixed coupling. I take this as the unperturbed potential. Its solution involves the usual Coulomb energies and wave functions. I regard the remaining terms as a perturbing potential, and evaluate their contribution using ordinary nonrelativistic time-independent perturbation theory.

To evaluate the contribution of the perturbing potential to the energy of the state with principal quantum number $n$, orbital quantum number $l$, and azimuthal quantum number $m$, one evaluates the matrix element

$$\Delta E_{nl} = \sum_{k=1}^{\infty} \langle \psi_{nlm} | V_k | \psi_{nlm} \rangle.$$  \hspace{1cm} (4.6)

Since the potential is radially symmetric, the states with different azimuthal quantum numbers are degenerate. For simplicity I momentarily restrict my attention to the
ground state. For quarks of mass $m_Q$, the ground-state wave function and energy are

$$
\psi_{100}(r) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}
$$

$$
E_1 = \frac{m_Q}{4} \left( \frac{4}{3} \alpha_s \right)^2
$$

where $a$ is the Bohr radius,

$$
a = \left( \frac{m_Q}{2} \frac{4}{3} \alpha_s \right)^{-1}.
$$

(The calculation for arbitrary $n, l, m$ is neither more difficult nor more enlightening; I present the result later.) Inserting Eq. (4.7) into Eq. (4.6), dividing by $E_1$, and integrating over angles yields

$$
\frac{\Delta E_{10}}{E_1} = \frac{8}{a^2} \sum_{k=1}^{\infty} (\beta \alpha_s)^k \int_0^\infty dr \ln^k \mu r e^{-2r/a}.
$$

Making the change of variables $z = \mu r$ and setting $c = \mu a/2$ gives

$$
\frac{\Delta E_{10}}{E_1} = \frac{2}{c^2} \sum_{k=1}^{\infty} (\beta \alpha_s)^k I_k
$$

where

$$
I_k = \int_0^\infty dz \ln^k z e^{-z/c} = \int_0^\infty dz z e^{k \ln z - z/c}
$$

(4.12)

For large $k$, the integral of Eq. (4.12) is dominated by the region where the argument of the exponential is a maximum. I denote this maximum by $z_0$, defined by the implicit equation

$$
z_0 \ln z_0 = kc.
$$

(4.13)

The integral may be approximated by the saddle-point method, which yields

$$
I_k \approx \sqrt{2\pi cz_0} z_0 \ln^k z_0 e^{-z_0/c}.
$$

(4.14)
Thus Eq. (4.11) becomes

$$\frac{\Delta E_{10}}{E_1} \approx \sqrt{8\pi} \sum_{k=1}^{\infty} \left( \frac{\tilde{z}_0}{c} \right)^{3/2} (\beta \alpha_s \ln z_0)^k e^{-z_0/z}. \tag{4.15}$$

The series for the energy correction given by Eq. (4.15) is divergent. For large \( k \), the size of the \( k \)th term is governed by the factor

$$S_k = (\beta \alpha_s \ln z_0)^k e^{-z_0/z}. \tag{4.16}$$

The term where the series begins to diverge is obtained by finding the minimum of this function with respect to \( k \) (keeping in mind that \( z_0 \) implicitly depends on \( k \) via Eq. (4.13)). Let \( \tilde{z}_0 \) be the value of \( z_0 \) that minimizes \( S_k \):

$$0 = \frac{d \ln S_k}{dk} = \ln(\beta \alpha_s) + \ln \ln \tilde{z}_0 + \left( \frac{k}{\tilde{z}_0 \ln \tilde{z}_0} - \frac{1}{c} \right) \left. \frac{dz_0}{dk} \right|_{z_0 = \tilde{z}_0}. \tag{4.17}$$

Using Eq. (4.13), the last two terms cancel. Solving for \( \tilde{z}_0 \) yields

$$\tilde{z}_0 = e^{1/\beta \alpha_s} = \frac{\mu}{\Lambda_{QCD}} \tag{4.18}$$

where the last relation follows from the definition of the QCD scale parameter \( \Lambda_{QCD} \):

$$\alpha_s \equiv \alpha_s(\mu) = \left( \beta \ln \frac{\mu}{\Lambda_{QCD}} \right)^{-1}. \tag{4.19}$$

Recalling that \( z = \mu r \), Eq. (4.18) implies that the series begins to diverge when the integral in Eq. (4.10) is dominated by distances of order \( \Lambda_{QCD}^{-1} \).

Within the context of perturbation theory, the best one can do is to truncate the series at the minimum term. The size of this term is an estimate of the inherent ambiguity in the perturbative calculation of the energy. Inserting Eq. (4.18) into
Eq. (4.15) yields

\[
\frac{\delta E_{10}}{E_1} \approx \sqrt{8\pi} \left( \frac{\varepsilon_0}{c} \right)^{3/2} e^{-\varepsilon_0/c} = \sqrt{8\pi} \left( \frac{2}{a\Lambda_{\text{QCD}}} \right)^{3/2} e^{-2a\Lambda_{\text{QCD}}} .
\]

(4.20)

The analogous calculation for the state with quantum numbers \( n, l, m \) with wavefunction\(^1\)

\[
\psi_{nlm}(r, \theta, \phi) = \frac{2}{n^2a^{3/2}} \left( \frac{(n-l-1)!}{(n+l)!} \right)^{1/2} \left( \frac{2\tau}{na} \right)^l L_{n-l-1}^{2l+1}(2\tau/na)e^{-\tau/na}Y^m_l(\theta, \phi)
\]

(4.21)

and energy

\[
E_n = -\frac{m_Q}{4\pi^2} \left( \frac{4}{3} \alpha_s \right)^2
\]

(4.22)

yields the result

\[
\frac{\delta E_{nl}}{E_n} \approx \sqrt{8\pi} \left( \frac{(n-l-1)!}{(n+l)!} \right)^{2l+2} \left( \frac{2}{na\Lambda_{\text{QCD}}} \right)^{2l+2} \left[ L_{n-l-1}^{2l+1}(2/na\Lambda_{\text{QCD}}) \right]^2 e^{-2a\Lambda_{\text{QCD}}}
\]

(4.23)

(with \( a \) given by Eq. (4.9)).

4.2 Ambiguities and Nonperturbative Contributions

In a full QCD calculation, the ambiguity in the perturbative calculation of the energy, Eq. (4.23), would be canceled by a related ambiguity in the nonperturbative contribution to the energy, resulting in an unambiguous expression. This cancelation between perturbative and nonperturbative ambiguities is in the same spirit as the analogous cancelation which occurs for infrared renormalons [7]. Thus I learn that quarkonia energies have a nonperturbative contribution which scales with \( na\Lambda_{\text{QCD}} \)

\(^1\)I use the notation and conventions of [46]. The \( L_{n-l-1}^{2l+1} \) are associated Laguerre polynomials.
like Eq. (4.23). However, it is impossible to calculate the coefficient of this nonperturbative contribution within the context of perturbation theory.

The ambiguity in the perturbative calculation of the energy, Eq. (4.23), is nearly proportional to the probability density that the quark and antiquark are separated by a distance of $\Lambda_{\text{QCD}}^{-1}$, which is given by $\rho^2 |\phi_{n^l,m^l}(\rho)|^2 \sim \rho^{2l+2} [L_{n-l-1}^{2l+1}(\rho)]^2 e^{-\rho}$, with $\rho = 2/na\Lambda_{\text{QCD}}$. The ambiguity increases as $\rho$ decreases, i.e., as the size of the state increases, until $\rho \sim O(n)$, at which point it apparently begins to decrease. However, recalling that the size of quarkonia is roughly $n^2 a$, one sees that this value of $\rho$ corresponds to quarkonia of size $\sim \Lambda_{\text{QCD}}^{-1}$, for which a perturbative approach is unreliable, and Eq. (4.23) should not be trusted.

The ambiguity in the perturbative calculation of the energy, Eq. (4.23), is proportional to $e^{-2/na\Lambda_{\text{QCD}}}$ rather than a power of $a\Lambda_{\text{QCD}}$ (as would arise from an infrared renormalon) because the series for the energy correction, Eq. (4.15), diverges less rapidly thanfactorially. An approximate solution to Eq. (4.13) is $z_0 \approx kc/\ln kc$, thus the factor $\ln^k z_0 e^{-z_0/c}$ in Eq. (4.15) is approximately $\ln^k e^{-k}$, which grows less rapidly thanfactorially, $k! \sim k^k e^{-k}$. A similar divergent series, resulting in a perturbative ambiguity proportional to $e^{-Q(1-\tau)/\Lambda_{\text{QCD}}}$, appears in the calculation of soft-gluon resummation in the production of an object of mass $Q$ from a hadron collision of total energy $\sqrt{S}$ ($\tau = Q^2/S$) [47].

A nonperturbative contribution to quarkonia energies from the gluon condensate has been previously studied [45]. This contribution is associated with an infrared renormalon arising from emission and absorption of soft gluons, as shown in Fig. 4.2.
Figure 4.2: A higher-order contribution to the quarkonium energy from the emission and absorption of gluons. The insertion of vacuum-polarization subdiagrams in the gluon propagator leads to an infrared renormalon associated with the gluon condensate.

The perturbative ambiguity from this renormalon is canceled by a related ambiguity in the gluon condensate. The gluon condensate is thought to give a contribution to the quarkonia energies of

$$\Delta E_{nl} = \frac{\pi}{16} n^2 q^6 a^4 < \alpha_s G^2 > \epsilon_{nl}$$

(4.24)

where $\epsilon_{nl}$ are numbers of order unity [45]. Since $< \alpha_s G^2 >$ is of order $\Lambda_{QCD}^4$, this contribution is proportional to $(a\Lambda_{QCD})^4$. Both the physical origin of Eq. (4.24) and its functional dependence on $a\Lambda_{QCD}$ show that it is a completely separate nonperturbative effect from the one considered in this paper.

The gluon condensate has been considered as the leading nonperturbative contribution to quarkonia in various analyses [48]. For states which are small compared with $\Lambda_{QCD}^{-1}$, the gluon condensate, proportional to $(a\Lambda_{QCD})^4$, is formally more important than the nonperturbative effect discussed in this paper, proportional to $e^{-2/n^2 a\Lambda_{QCD}}$. However, for larger states where $n^2 a$ approaches $\Lambda_{QCD}^{-1}$, the relative size of the gluon-condensate contribution and the nonperturbative contribution considered in this chapter are unknown, the normalization of the latter is not known. It
is important to establish the size of this nonperturbative effect in order to judge its impact on quarkonia energies.
Chapter 5

Conclusions

5.1 QCD and Yukawa Corrections to $q\bar{q} \rightarrow t\bar{b}$

The best way to directly measure the Standard Model parameter $V_{tb}$ is via single-top-quark production. An accurate calculation of the production cross section is necessary in order to extract $|V_{tb}|$ from experiment. I have calculated the next-to-leading-order cross section for the process $q\bar{q} \rightarrow t\bar{b}$ with a top-quark mass of 175 GeV to be $0.88 \pm 0.05$ pb at the Fermilab Tevatron and $10.2 \pm 0.6$ pb at the CERN Large Hadron Collider (LHC). (The errors do not include uncertainties in the top-quark mass). The size of the QCD correction is significant, +54% at the Tevatron and +50% at the LHC, and improves the outlook for observation of this process during Run II at the Tevatron.

Much can be done in the future to reduce the uncertainty in this calculation. The next-to-next-to-leading order correction to the Drell-Yan process is already known [14]. The full next-to-next-to-leading order QCD correction to $q\bar{q} \rightarrow t\bar{b}$ should be completed in the near future. This should reduce the uncertainty in the cross section from uncalculated higher orders to below the 1% level. Uncertainties due to the
parton distribution functions (PDFs) are likely to be of the same order, although a reliable estimate of such an uncertainty will have to wait for until a set of PDFs with built-in uncertainties is available. Such a set is expected in the near future.

It seems likely that by the time the process $q\bar{q} \rightarrow t\bar{b}$ is observed in Run II at the Tevatron, the theoretical uncertainty in the cross section will be slightly larger than $\pm 10\%$, due mostly to the uncertainty in the mass. This is adequate in comparison with the anticipated experimental errors. The statistical error on the measured cross section in Run II will be about $\pm 20\%$ [10]. This corresponds to a measurement of $|V_{tb}|$ with an accuracy of $\pm 10\%$ (assuming $|V_{tb}| \approx 1$). A high-luminosity Tevatron, which could potentially deliver 30 fb$^{-1}$ over several years, would allow a measurement of the cross section with a statistical uncertainty of about 3%, with a comparable theoretical uncertainty. Combining the statistical and theoretical uncertainties in quadrature, this corresponds to a measurement of $|V_{tb}|$ with an accuracy of about $\pm 4\%$.

5.2 Top-Quark Pole Mass

The confinement of color, a nonperturbative property of QCD, precludes the existence of S-matrix poles at quark masses and impedes any attempt to unambiguously define the pole mass of a stable heavy quark. The same is true of the top-quark pole mass despite the fact that the top-quark width is much greater than the strong-interaction energy scale, $\Lambda_{QCD}$. This is signaled by the divergent behavior at large orders of the expansion of the top-quark self-energy in powers of $\alpha = \beta_0 \alpha_s(m_t)/4\pi$, which leads to an unavoidable ambiguity of $O(\Lambda_{QCD})$ in the top-quark pole mass. The
top-quark width does not suffer from an ambiguity of the same order. The ambiguity in the pole mass does not limit the accuracy with which a short-distance mass, such as the \( \overline{\text{MS}} \) mass, can be measured.

5.3 Nonperturbative Effects in Quarkonia

The calculation of quarkonia energies is divergent at large orders in perturbation theory. The associated ambiguity in the perturbative result is proportional to \( e^{-1/na}\Lambda_{\text{QCD}} \), where \( n \) is the principal quantum number, \( a \) is the Bohr radius, and \( \Lambda_{\text{QCD}} \) is the QCD scale parameter. This ambiguity is associated with a nonperturbative contribution to the energy, coming from distances of order \( \Lambda_{\text{QCD}}^{-1} \) and greater. This is a separate nonperturbative effect from the gluon-condensate contribution to quarkonia energies.
Bibliography


[31] $\mu^+\mu^-$ Collider: A Feasibility Study, $\mu^+\mu^-$ Collider Collaboration, BNL-52503 (1995).


Curriculum Vitae

The author was born in West Allis, Wisconsin on December 7, 1967. He received a Bachelor of Arts degree from Northwestern University in 1992 and a Master of Science degree from the University of Illinois at Chicago in 1993. He attended the University of Illinois at Urbana-Champaign from 1993–1998, pursuing research in high-energy physics under the direction of Professor Scott Willenbrock and working as a teaching assistant. From 1995–1997, he held a GAANN fellowship from the U.S. Department of Education.